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Torsion-Free Metabelian Groups with Commutator Quotient $C_{p^n} \times C_{p^m}$

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1. INTRODUCTION

Let G be a finitely generated torsion-free metabelian group with finite commutator quotient. Then G is a Bieberbach group; that is, G is a torsion-free group containing a normal, maximal abelian subgroup V of finite rank and index. The subgroup V and the quotient G/V are known as the *translation subgroup* and the *point-group* (or *holonomy group*) of G, respectively. It is well known that the finiteness of the commutator quotient of G is equivalent to the triviality of the centre of G [6]. In Theorem A of [3], we showed that every Bieberbach group with finite commutator quotient and point-group isomorphic to $C_{p^n} \times C_{p^m}$ contains a subgroup isomorphic to a torsion-free quotient of

$$K(p^n, p^m) = \langle a, b \mid (a^{p^n})^{t(p^m, b)}, (b^{p^m})^{t(p^n, a)}, [[a, b], a^{p^n}],$$
$$[[a, b], b^{p^m}], \textit{metabelian} \rangle,$$

where $t(s, x) = \sum_{i=0}^{s-1} x^i$ and the presentation is written relative to the variety of metabelian groups. Furthermore, we showed that $K(p^n, p^m)$ is itself a Bieberbach group of dimension $p^{n+m} - 1$, with point-group $C_{p^n} \times C_{p^m}$ and commutator quotient $C_{p^{n+m}} \times C_{p^{n+m}}$.

In [5], Gupta and Sidki study the existence of torsion-free metabelian groups with a finite elementary abelian p-group as commutator quotient. In particular, they showed that $K(p, p)$ has no proper torsion-free quotients and proved the following theorem.

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THEOREM 2 OF [5]. *Let G be a metabelian group such that G/G' is a finite p-group for some prime p. Suppose furthermore that H is a subgroup of G such that $G = G'H$. Then $H' = G' \cap H$.*

They applied the theorem above and the fact that $K(p, p)$ has no proper torsion-free quotients to show that a finitely generated torsion-free metabelian group cannot have commutator quotient isomorphic to $C_p \times C_p$, p prime [5]. On working with the torsion-free quotients of $K(p^n, p^m)$, we are able to investigate the possibilities for a 2-generated abelian p-group to be the commutator quotient of a finitely generated torsion-free metabelian group. In Section 2 we introduce the tools in order to study such quotients. In Section 3, considering the quotients of $K(p, p^m)$, we prove

THEOREM A. *There exists a finitely generated torsion-free metabelian group G with commutator quotient isomorphic to $C_{p^n} \times C_{p^m}$ if and only if $n, m \geq 2$.*

In Section 4 we describe the calculations to obtain the torsion-free quotients of $K(p, p^2)$. Furthermore, we present the results obtained in [4] for the groups $K(2, 8)$ and $K(4, 4)$. Using the list of torsion-free quotients of $K(4, 4)$ we obtain

THEOREM B. *Let G be a finitely generated, torsion-free metabelian group, with commutator quotient isomorphic to $C_4 \times C_4$. Then G is isomorphic to*

$$K(2, 2) = \langle a, b \mid (a^2)^{1+b}, (b^2)^{1+a}, [[a, b], a^2], [[a, b], b^2],\ \textit{metabelian}\rangle,$$

the fundamental group of the Hantzsche–Wendt manifold.

2. THE GROUP $K(p^n, p^m)$

We recall the notation introduced in [3]. Let

$$F_n = \langle x_1, \ldots, x_n |\ \textit{metabelian}\rangle$$

denote the free group of rank n in the variety of metabelian groups. A finitely generated metabelian group G is presented as

$$G = \langle x_1, \ldots, x_n | R_1, R_2, \ldots, R_s,\ \textit{metabelian}\rangle \cong F_n/\langle R_1, R_2, \ldots, R_s\rangle^{F_n}.$$

We define the following polynomials, for $s \in \mathbb{N}$:

$$t(s, x) = 1 + x + \cdots + x^{s-1}$$

$$d(x) = x - 1$$

$$l(s, x) = (t(s, x) - s)/d(x) = \sum_{i=1}^{s-1} t(i, x) = \sum_{i=0}^{s-2} (s - i - 1)x^i.$$

If $g, x_1, \dots, x_n$ are elements of group G, and $s_1, \dots, s_n \in \mathbb{Z}$, then we write

$$g^{s_1x_1+s_2x_2+\cdots+s_nx_n}$$

for the element $(g^{s_1})^{x_1}(g^{s_2})^{x_2}\cdots(g^{s_n})^{x_n}$.

Whenever it is convenient, we will write additively in abelian subgroups of G. When the commutator subgroup G' of G is abelian, using the module notation, we write

$$[x_1, x_2^s] = [x_1, x_2].t(s, x_2).$$

Consider then

$$K(p^n, p^m) = \langle a, b \mid (a^{p^n})^{t(p^m, b)}, (b^{p^m})^{t(p^n, a)}, [[a, b], a^{p^n}], \\ [[a, b], b^{p^m}],\ metabelian\rangle.$$

We recall that the group $G = K(p^n, p^m)$ is a Bieberbach group of dimension $p^{n+m} - 1$, with point-group isomorphic to $C_{p^n} \times C_{p^m}$ and commutator quotient $C_{p^{n+m}} \times C_{p^{n+m}}$. The commutator subgroup G' of G is free abelian of rank $p^{n+m} - 1$, and if we denote the commutator $[a, b]$ by c and the action of a and b on G' by A and B, respectively, it follows that G' is freely generated by the set

$$\{c.A^iB^j, 0 \le i < p^n, 0 \le j < p^m, (i, j) \neq (p^n - 1, p^m - 1)\}.$$

Furthermore $V = \langle a^{p^n}, b^{p^m}, G'\rangle$ is the translation subgroup of G.

LEMMA 2.1. *Let M be the $\mathbb{Q}[\frac{G}{V}]$-module defined as $M = \mathbb{Q} \otimes V$. Then M decomposes as a direct sum of*

$$(m - n)p^n + (p + 1)\frac{p^n - 1}{p - 1}$$

irreducible, non-isomorphic submodules.

Proof. It is clear that as $\mathbb{Q}[\frac{G}{V}]$-module, M is cyclic and it is generated by c. And since for $s \ge 1$, we have $\gcd(d(x), t(p^s, x)) = 1$, we are able to write

$$M = M_1 \oplus M_2 \oplus M_3 \oplus M_4,$$

where

$$M_1 = M.d(A)d(B), \qquad M_2 = M.t(p^n, A)d(B)$$
$$M_3 = M.d(A)t(p^m, B), \qquad M_4 = M.t(p^n, A)t(p^m, B).$$

Furthermore we have $M.t(p^n, A)d(A) = M.t(p^m, B)d(B) = 0$. Thus the submodule M_4 is central in G and is therefore trivial. When $s \ge 2$, the polynomial $t(p^s, x)$ can be factored as $t(p^{s-i}, x)t(p^i, x^{p^{s-i}})$, for $1 \leq i \leq s-1$. Thus we can write

$$t(p^s, x) = t(p, x)t(p, x^p)t(p, x^{p^2}) \cdots t(p, x^{p^{s-1}}),$$

where all the terms are irreducible over $\mathbb{Q}$. Let U_j be the companion matrix of the polynomial $t(p, x^{p^{j-1}})$ and let Id be the identity matrix. Since M is generated by c, we are able to find a basis for M_2 such that $[A] = Id$ and

$$B = \begin{pmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_m \end{pmatrix}.$$

Similarly, there exists a basis of M_3 such that $[B] = Id$ and

$$A = \begin{pmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_n \end{pmatrix}.$$

Therefore M_2 and M_3 decompose as

$$M_2 = \bigoplus_{j=1}^{m} M_{2j} \qquad \text{and} \qquad M_3 = \bigoplus_{j=1}^{n} M_{3j},$$

where the submodules M_{2j} and M_{3j} have dimension $p^{j-1}(p-1)$. The actions of a and b on these submodules are given by the matrices above.

On M_1, we have that A and B have $t(p^n, x)$ and $t(p^m, x)$ as minimal polynomials, respectively. If we extend the field of rationals $\mathbb{Q}$ by B, we obtain the algebra

$$\mathbb{Q}[B] \cong \bigoplus_{j=1}^{m} \mathbb{Q}[U_j].$$

And if we extend the algebra $\mathbb{Q}[B]$ by A, we have

$$\mathbb{Q}[B][A] \cong \bigoplus_{j=1}^{m} \mathbb{Q}[U_j][A] \cong \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{n} \mathbb{Q}[U_j^B][U_i^A].$$

Now we can verify in a straightforward manner that these submodules decompose as direct sum of irreducible submodules. Furthermore, it should be clear that they are all non-isomorphic. And it follows from Proposition 2.6 of [7], which describes the structure of the algebra $\mathbb{Q}[\frac{G}{V}]$, that the number of irreducible submodules of M is equal to the number of non-trivial cyclic subgroups of $C_{p^n} \times C_{p^m}$. By induction on $(m+n)$, we can show that $C_{p^n} \times C_{p^m}$ has

$$(m-n)p^n + (p+1)\frac{p^n - 1}{p-1}$$

non-trivial cyclic subgroups, and the result follows. ∎

Notice that we have $(b^{p^m})^{d(b)} = [b^{p^m}, b] = e = [a^{p^n}, a] = (a^{p^n})^{d(a)}$. Now, since $\ker(d(B)) = M_3$ and $\ker(d(A)) = M_2$, we have

$$b^{p^m} \in M_3 \qquad \text{and} \qquad a^{p^n} \in M_2.$$

LEMMA 2.2. *Let G be a Bieberbach group with translation subgroup V. Furthermore let $N_1, N_2 \trianglelefteq G$, such that G/N_1 and G/N_2 are both torsion-free. If $\mathbb{Q} \otimes (N_1 \cap V) \subseteq \mathbb{Q} \otimes (N_2 \cap V)$, then $N_1 \leq N_2$.*

Proof. We denote $\mathbb{Q} \otimes (N_i \cap V)$ by R_i. Since G/N_1 and G/N_2 are torsion-free, $N_1 \cap V$ and $N_2 \cap V$ are both pure submodules of V and

$$N_1 \cap V = R_1 \cap V \subseteq R_2 \cap V = N_2 \cap V.$$

Let $[G : V] = n$. If $x_1 \in N_1$, then $x_1^n \in N_1 \cap V \subseteq N_2 \cap V$. Since G/N_2 is torsion-free and $x_1^n \in N_2$, we must have $x_1 \in N_2$ and $N_1 \leq N_2$. ∎

We describe now the method we use to compute the torsion-free quotients of $K(p^n, p^m)$. Let N be a non-trivial normal subgroup of $K(p^n, p^m)$. Then the module $R = \mathbb{Q} \otimes (N \cap V)$ is a non-trivial submodule of M. Since M is direct sum of

$$(m-n)p^n + (p+1)\frac{p^n - 1}{p-1} = k$$

irreducible, non-isomorphic submodules, it follows from the Krull–Schmidt theorem that R is equal to the sum of some of them. Thus we have $2^k - 1$ cases for R to study (we exclude the trivial one).

Suppose that, for a certain possibility for R, we find $N \trianglelefteq K(p^n, p^m)$ and $x \in K(p^n, p^m)$, such that $R = \mathbb{Q} \otimes (N \cap V)$ and $x \notin N$, but with $x^s \in N$, $s \geq 2$. Then $K(p^n, p^m)/N$ is not torsion-free but we can define $\overline{N}$ as the normal closure on $K(p^n, p^m)$ of the subgroup $\langle N, x\rangle$ and repeat the analysis with the subgroup $\overline{N}$. It is clear that we might have $\overline{R} = \mathbb{Q} \otimes (\overline{N} \cap V)$ different from R. Also, if x is one of the generators of $K(p^n, p^m)$, then the group $K(p^n, p^m)/\overline{N}$ is cyclic and finite. For instance, we have seen that $a^{p^n} \in M_2$ and $b^{p^m} \in M_3$. Therefore, neither M_2 nor M_3 can be contained in R, in order to obtain a torsion-free quotient. We should look for powers of the generators to eliminate some possibilities for R. Furthermore, it follows from Lemma 2.2 that, for any possibility for R being analysed, there will be at most one possible $N \trianglelefteq K(p^n, p^m)$, such that $\mathbb{Q} \otimes (N \cap V) = R$ and $K(p^n, p^m)/N$ is torsion-free.

If we denote by $\Lambda_{p,n,m}$ the set of representatives of isomorphism types of torsion-free quotients of $K(p^n, p^m)$, we can turn $\Lambda_{p,n,m}$ into a partially ordered set if we define for any $Q_1, Q_2 \in \Lambda_{p,n,m}$,

$$Q_1 \geq Q_2 \iff \exists N \trianglelefteq Q_1 \quad \text{s.t. } \frac{Q_1}{N} \cong Q_2.$$

Using this method, we compute in Section 4 the list of torsion-free quotients for the groups $K(p, p^2)$, $K(2, 8)$, and $K(4, 4)$, presenting the lattice of $\Lambda_{p,n,m}$ for the last two cases. In Section 3 we use the torsion-free quotients of $K(p, p^m)$ to obtain some general properties of torsion-free metabelian groups with finite commutator quotient. The problem of extending this method to the general case is due to the exponential growth of the possibilities of the $K(p^n, p^m)$-module $R = \mathbb{Q} \otimes (N \cap V)$.

3. QUOTIENTS OF $K(p, p^m)$

As in the previous section, let V be the translation subgroup of $K(p, p^m)$ and let U_j be the companion matrix of the polynomial $t(p, x^{p^{j-1}})$. We have seen in Lemma 2.1 that $M = \mathbb{Q} \otimes V$ decomposes as a direct sum of $mp + 1$ irreducible, non-isomorphic submodules

$$M = \bigoplus_{i=1}^{p-1} \bigoplus_{j=1}^{m} M_{1j_i} \bigoplus_{j=1}^{m} M_{2j} \bigoplus M_3,$$

where M_{1j_i} has dimension $p^{j-1}(p-1)$, with $[A] = U_j^{ip^{j-1}}$ and $[B] = U_j$. M_{2j} has dimension $p^{j-1}(p-1)$, with $[A] = Id$ and $[B] = U_j$, and M_3 has dimension $p-1$, where $[A] = U_1$ and $[B] = Id$.

LEMMA 3.1. *Following the terminology above, we have that*

$$(ab^k)^{p^m} \in M_{11_i},$$

for $1 \leq i \leq p-1$ *and* $k + i = p^m$, *and*

$$(ab^{kp^{j-1}})^{p^m} \in M_{21} \oplus \cdots \oplus M_{2(j-1)} \oplus M_{1j_i},$$

for $1 \leq i \leq p-1$, $2 \leq j \leq m$, *and* $k + i = p^{m-j+1}$.

Proof. We will show that $((ab^k)^{p^m})^{(a-b^r)} = e$ if $k + r = p^m$. Since both $(ab^k)^{p^m}$ and b^{p^m} are contained in V, they must commute. Thus $(ab^k)^{p^m}$ commutes with

$$b^{p^m}(ab^k)^{-1} = b^{p^m-k}a^{-1} = b^r a^{-1},$$

and we have

$$((ab^k)^{p^m})^{(1-b^r a^{-1})} = e.$$

We can conjugate the above expression by a, and we obtain

$$((ab^k)^{p^m})^{(a-b^r)} = e$$

if $k + r = p^m$.

Now let $r = ip^{j-1}$, where $1 \le i \le p-1$. By the decomposition we obtained for M, we have

$$\ker(A - B^{ip^{j-1}}) = M_{21} \oplus \cdots \oplus M_{2(j-1)} \oplus M_{1j_i}$$

when $2 \le j \le m$, and

$$\ker(A - B^i) = M_{11_i}$$

when $j = 1$. In fact, A acts as $B^{ip^{j-1}}$ on M_{1j_i} and as Id on M_{2s}, $1 \le s \le m$. Furthermore, B acts as the companion matrix of $t(p, x^{p^{s-1}})$ on M_{2s}. Therefore, for $1 \le s \le j-1$, $B^{ip^{j-1}}$ also acts as Id.

Thus we have

$$(ab^k)^{p^m} \in \ker(A - B^i) = M_{11_i}$$

for $1 \le i \le p-1$ and $k + i = p^m$, and

$$(ab^{kp^{j-1}})^{p^m} \in \ker(A - B^{ip^{j-1}}) = M_{21} \oplus \cdots \oplus M_{2(j-1)} \oplus M_{1j_i}$$

for $1 \le i \le p-1$, $2 \le j \le m$, and $k + i = p^{m-j+1}$. ∎

Remark. Notice that from the factorization of the polynomial $t(p^s, x)$ as

$$t(p^s, x) = t(p^{s-i}, x)t(p^i, x^{p^{s-i}}),$$

we can conclude that the group $K(p^n, p^m)$ has a torsion-free quotient isomorphic to $K(p^{n'}, p^{m'})$, for any $1 \le n' \le n$ and $1 \le m' \le m$.

PROPOSITION 3.2. *For any $2 \le i$, $j \le m+1$, the group $K(p, p^m)$ has a torsion-free quotient with commutator quotient isomorphic to $C_{p^i} \times C_{p^j}$.*

Proof. We use induction on m. If $m = 1$, then $i = j = 2$ and the proposition is true, since $K(p, p)$ itself has commutator quotient isomorphic to $C_{p^2} \times C_{p^2}$.

Let $m \ge 2$. Since $K(p, p^m)$ has a torsion-free quotient isomorphic to $K(p, p^{m-1})$, by induction we have that $K(p, p^m)$ has a torsion-free quotient with commutator quotient isomorphic to $C_{p^i} \times C_{p^j}$, for all $2 \le i$, $j \le m$. Because the commutator quotient of $K(p, p^m)$ is isomorphic to

$C_{p^{m+1}} \times C_{p^{m+1}}$, we have only to find $N_k \trianglelefteq K(p, p^m)$, such that $K(p, p^m)/N_k$ is torsion-free with commutator quotient $C_{p^k} \times C_{p^{m+1}}$, for $2 \le k \le m$.

For $2 \le k \le m$, let N_k be the normal closure on $K(p, p^m)$ of the subgroup generated by

$$(a^p)^{t(p^{k-1}, b^{p^{m+1-k}})}.$$

It is clear that N_k is contained in the translation subgroup V of $K(p, p^m)$, and $K(p, p^m)/N_k$ has commutator quotient $C_{p^k} \times C_{p^{m+1}}$. Then it remains to show that $K(p, p^m)/N_k$ is torsion-free.

We have seen that $a^p \in M_2$ and that B acts as the companion matrix of $t(p, x^{p^{j-1}})$ on M_{2j}. Since $t(p^{k-1}, b^{p^{m+1-k}})$ can be factored as

$$t\big(p^{k-1}, b^{p^{m+1-k}}\big) = t\big(p, b^{p^{m+1-k}}\big) \cdots t\big(p, b^{p^{m-2}}\big) t\big(p, b^{p^{m-1}}\big),$$

we have

$$\ker\big(t\big(p^{k-1}, B^{p^{m+1-k}}\big)\big) = M_{2(m+2-k)} \oplus \cdots \oplus M_{2m},$$

and the element $(a^p)^{t(p^{k-1}, b^{p^{m+1-k}})}$ is contained in $M_{21} \oplus M_{22} \oplus \cdots \oplus M_{2(m+1-k)}$, with non-trivial components in all these submodules. Therefore

$$\mathbb{Q} \otimes N_k = M_{21} \oplus M_{22} \oplus \cdots \oplus M_{2(m+1-k)}$$

and N_k has rank $\sum_{i=0}^{m-k} p^i(p-1) = p^{m+1-k} - 1$.

Then consider

$$\begin{aligned}
(a^p)^{t(p^{k-1}, b^{p^{m+1-k}})} &= a^p (a^p)^{b^{p^{m+1-k}}} \cdots (a^p)^{(b^{p^{m+1-k}})^{p^{k-1}-1}} \\
&= p^{k-1} a^p + c.t(p, A) t(p^{m+1-k}, B) l\big(p^{k-1}, B^{p^{m+1-k}}\big) \\
&= p^{k-1} a^p + c.(1 + \cdots + A^{p-1})\big(1 + \cdots + B^{p^{m+1-k}-1}\big) \\
&\quad \times \big((p^{k-1} - 1) + \cdots + (B^{p^{m+1-k}})^{p^{k-1}-2}\big).
\end{aligned}$$

It is clear that the set

$$\big\{(a^p)^{t(p^{k-1}, b^{p^{m+1-k}}) b^i}, 0 \le i \le p^{m+1-k} - 2\big\}$$

is a basis of N_k. Therefore the elements of N_k can be expressed as

$$(a^p)^{t(p^{k-1}, b^{p^{m+1-k}}) f(b)},$$

where $f(b) \in \mathbb{Z}[b]$, of degree at most $p^{m+1-k} - 2$. If we compute the Smith normal form for the matrix of generators of V/N_k, we can verify in a straightforward manner that V/N_k is torsion-free. We illustrate these calculations with the group $K(2, 8)$ and with N_2 being the normal closure on $K(2, 8)$ of the subgroup generated by $(a^2)^{t(2, b^4)} = (a^2)^{1+b^4}$.

The subgroup N_2 is abelian of rank 3, with free generators

$$(a^2)^{1+b^4}, (a^2)^{b+b^5}, (a^2)^{b^2+b^6}.$$

If we write the elements above in terms of the basis of V and construct the matrix of generators of V/N_2, we get

$$\begin{pmatrix} 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Notice that the last non-zero entry in the last row is equal to 1 and is contained in a column that has all other entries equal to zero. Therefore we can perform elementary column operations and obtain a new matrix, whose last row has only one non-zero entry, which is equal to 1, with all the other rows remaining unchanged. Then we can repeat this procedure with the other rows, until we reach a matrix, equivalent to the above, of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus V/N_2 is torsion-free. The general case is similar to the above, with the rows of the matrix of generators of V/N_k presenting the same characteristics as of the one above, which allows us in the same manner to conclude that V/N_k is torsion-free. Therefore, to show that $K(p, p^m)/N_k$ is torsion-free, it remains to show that there exists no $g \in K(p, p^m)\backslash V$, such that $g^{p^m} \in N_k$.

We should recall that the elements $p^m a^p$ and pb^{pm} are contained in the commutator subgroup of $K(p, p^m)$, and therefore these can be expressed in terms of the basis of $K(p, p^m)'$. Indeed, we have $p^m a^p = -c.t(p, A)l(p^m, B)$ and $pb^{pm} = c.t(p^m, B)l(p, A)$, and we can write

$$c.A^{p-1}B^{p^m-2} = -p^m a^p - c.\big(t(p, A)l(p^m, B) - A^{p-1}B^{p^m-2}\big)$$

and

$$c.A^{p-2}B^{p^m-1} = pb^{p^m} - c.\big(t(p^m, B)l(p, A) - A^{p-2}B^{p^m-1}\big).$$

Every $g \in K(p, p^m)$ can be written as $g = a^i b^j v$, where $0 \le i \le p-1$, $0 \le j \le p^m - 1$, and $v \in V$. Then

$$\begin{aligned} g^{pm} &= (a^i b^j v)^{p^m} \\ &= ip^{m-1}a^p + jb^{p^m} - c.t(j, B)t(i, A)\sum_{k=1}^{p^m-1} t(k, A^i)B^{jk} + v.t(p^m, A^i B^j) \end{aligned}$$

and we should verify if the equation

$$ip^{m-1}a^p + jb^{p^m} - c.t(j,B)t(i,A)\sum_{k=1}^{p^m-1} t(k,A^i)B^{jk} + v.t(p^m, A^iB^j)$$
$$= p^{k-1}(a^p)^{f(b)} + c.t(p,A)t(p^{m+1-k},B)l(p^{k-1},B^{p^{m+1-k}})f(B)$$

has non-trivial solutions. Since $f(b)$ has degree at most $p^{m+1-k}-2$, the term in $(a^p)^{t(p^{k-1},b^{p^{m+1-k}})f(b)}$ with the highest sum of exponents would be $c.A^{p-1}B^{p^m-3}$, and therefore the term b^{p^m} will not appear in the expression

$$p^{k-1}(a^p)^{f(b)} + c.t(p,A)t(p^{m+1-k},B)l(p^{k-1},B^{p^{m+1-k}})f(B).$$

Then p must divide j, since if the term b^{p^m} appears in the expression

$$-c.t(j,B)t(i,A)\sum_{k=1}^{p^m-1} t(k,A^i)B^{jk} + v.t(p^m, A^iB^j),$$

its coefficient would be a multiple of p. Therefore $g = a^ib^{j'p}v$ and

$$g^{p^{m-1}} = ip^{m-2}a^p + j'b^{p^m} - c.t(j'p,B)t(i,A)\sum_{k=1}^{p^{m-1}-1} t(k,A^i)B^{j'kp}$$
$$+ v.t(p^{m-1}, A^iB^{j'p}) \in V.$$

We can repeat the argument above m times and conclude that p^m divides j. Then $g = a^iv'$, with $v' \in V$, and $g^p \in V$. Thus the equation can be written for g^p as

$$g^p = ia^p + v'.t(p,A^i)$$
$$= p^{k-1}(a^p)^{f(b)} + c.t(p,A)t(p^{m+1-k},B)l(p^{k-1},B^{p^{m+1-k}})f(B),$$

and using the same argument, now with a^p, we finally conclude that p divides i and thus arrive at $g \in V$. Thus $K(p,p^m)/N_k$ is torsion-free, of dimension

$$rk(V) - rk(N_k) = p^{m+1} - 1 - (p^{m+1-k} - 1)$$
$$= p^{m+1} - p^{m+1-k} = p^{m+1-k}(p^k - 1).$$

The quotient $K(p,p^m)/N_k$ has also point-group isomorphic to $C_p \times C_{p^m}$, since it is not isomorphic to a quotient of $K(p,p^{m-1})$. ∎

Remark. We have seen that the group $K(p^n, p^m)$ has a torsion-free quotient isomorphic to $K(p^{n'}, p^{m'})$, for any $1 \leq n' \leq n$ and $1 \leq m' \leq m$. In particular, when working with the group $K(p, p^m)$, we have that if $N_{m'}$ is the normal closure on $K(p, p^m)$ of the subgroup

$$\left\langle (a^p)^{t(p^{m'}, b)}, \left(b^{p^{m'}}\right)^{t(p, a)}, \left[c, b^{p^{m'}}\right]\right\rangle,$$

then $K(p, p^m)/N_{m'} \cong K(p, p^{m'})$. In this case, for $1 \leq m' \leq m-1$, we have

$$R_{m'} = \mathbb{Q} \otimes (N_{m'} \cap V) = \bigoplus_{i=1}^{p-1} \bigoplus_{j=m'+1}^{m} M_{1j_i} \bigoplus_{j=m'+1}^{m} M_{2j}.$$

PROPOSITION 3.3. *The group $K(p, p^m)$ has no torsion-free quotient with commutator quotient isomorphic to $C_p \times C_{p^m}$.*

Proof. Let V be the translation subgroup of $K(p, p^m)$, let M be the module $\mathbb{Q} \otimes V$, and let N be a non-trivial normal subgroup of $K(p, p^m)$. Then $R = \mathbb{Q} \otimes (N \cap V)$ is a non-trivial submodule of M, and it should be the sum of some of the $mp + 1$ submodules obtained in the decomposition of M. Suppose that $K(p, p^m)/N$ is torsion-free. It follows from Lemma 3.1 that $M_3, M_{11_i} \not\subseteq R$. We divide the possibilities for R in 2 cases.

First suppose that $M_{21} \not\subseteq R$. Then it follows from Lemma 2.2 and the previous remark that $K(p, p^m)/N$ has a torsion-free quotient isomorphic to $K(p, p)$. Since $K(p, p)$ has commutator quotient isomorphic to $C_{p^2} \times C_{p^2}$, it is clear that $K(p, p^m)/N$ cannot have commutator quotient isomorphic to $C_p \times C_{p^m}$.

Suppose now that $M_{21} \subseteq R$. If $m = 1$, then $K(p, p^m)/N$ is not torsion-free, since $a^p \in M_{21}$. Consider then $m \geq 2$. We ask which of the submodules M_{12_i}, M_{22} are contained in R. It follows from Lemma 3.1 that M_{12_i} cannot be contained in R, since $(ab^{kp})^{p^m} \in M_{21} \oplus M_{12_i}$ for $1 \leq i \leq p-1$ and $k + i = p^{m-1}$. If $M_{22} \subseteq R$, we repeat this analysis, this time with the submodules M_{13_i}, M_{23}, and so on. Since $a^p \in M_2$, there exists $2 \leq s \leq m$, such that $M_{21} \oplus \cdots \oplus M_{2(s-1)} \subseteq R$, and $M_{1s_i}, M_{2s} \not\subseteq R$, for $1 \leq i \leq p-1$.

Now we apply Proposition 3.2 to the group $K(p, p^s)$. If N_2 is the normal closure on $K(p, p^s)$ of the subgroup generated by $(a^p)^{t(p, b^{p^{s-1}})}$, then $H = K(p, p^s)/N_2$ is torsion-free and has commutator quotient isomorphic to $C_{p^2} \times C_{p^{s+1}}$. However, it follows again from Lemma 2.2 and the previous remark that $K(p, p^m)/N$ has a torsion-free quotient isomorphic to H and therefore cannot have commutator quotient $C_p \times C_{p^m}$. ∎

We are now able to prove Theorem A.

THEOREM A. *There exists a finitely generated torsion-free metabelian group G with commutator quotient isomorphic to $C_{p^n} \times C_{p^m}$ if and only if $n, m \geq 2$.*

Proof. By the result of Proposition 3.2, it remains to show that there is no finitely generated, torsion-free metabelian group with commutator quotient isomorphic to $C_p \times C_{p^m}$, for $m \geq 1$. Suppose that there exists a metabelian group of this type. If $x, y \in G$ are the generators of G modulo G' and $H = \langle x, y \rangle$, then it follows from Theorem 2 of [5] that H is a 2-generated, metabelian Bieberbach group, with

$$\frac{H}{H'} \cong \frac{G}{G'} \cong C_p \times C_{p^m}.$$

Furthermore, if we denote by V_H the translation subgroup of H, we have that $H' \leq V_H$, and Theorem A of [3] tells us that H is isomorphic to a torsion-free quotient of $K(p, p^m)$. However, it follows from the previous proposition that $K(p, p^m)$ does not have a torsion-free quotient of this type, and we reach a contradiction. ∎

We now compute the torsion-free quotients for some other groups $K(p^n, p^m)$, using the method described in Section 2. We illustrate this method with the calculations for $K(p, p^2)$. In [4], one can find the calculations for $K(2, 8)$ and $K(4, 4)$.

4. TORSION-FREE QUOTIENTS OF $K(p, p^2)$

Let

$$K(p, p^2) = \big\langle a, b \mid (a^p)^{t(p^2, b)}, (b^{p^2})^{t(p, a)}, [[a, b], a^p], \\ \big[[a, b], b^{p^2}\big],\ \textit{metabelian}\big\rangle.$$

The group $K(p, p^2)$ is a Bieberbach group of dimension $p^3 - 1$, with point-group isomorphic to $C_p \times C_{p^2}$ and commutator quotient isomorphic to $C_{p^3} \times C_{p^3}$. Let V denote once more the translation subgroup of $G = K(p, p^2)$ and $c = [a, b]$. It follows from Section 2 that the module $M = \mathbb{Q} \otimes V$ decomposes as a sum of $2p + 1$ irreducible, non-isomorphic submodules

$$M = \bigoplus_{i=1}^{p-1} \bigoplus_{j=1}^{2} M_{1j_i} \bigoplus_{j=1}^{2} M_{2j} \bigoplus M_3,$$

where M_{11_i}, M_{21}, M_3 have dimension $p - 1$ and M_{12_i}, M_{22} have dimension $p(p - 1)$. The actions of a and b on these submodules were described in the previous section.

We have that $b^{p^2} \in M_3$ and $a^p \in M_2$. Furthermore, $(a^p)^{t(p, b^p)} \in M_{21}$ and $(a^p)^{t(p, b)} \in M_{22}$, and both are non-trivial. It follows from Lemma 3.1 that

$$(ab^k)^{p^2} \in M_{11_i}$$

for $1 \le i \le p-1$ and $k+i = p^2$, and

$$(ab^{kp})^{p^2} \in M_{21} \oplus M_{12_i}$$

for $1 \le i \le p-1$ and $k+i = p$. For the last case, we have

$$\begin{aligned}((ab^{kp})^p)^{t(p,\, b^p)} &= (a^p)^{t(p,\, b^p)} + kpb^{p^2} - c.t(kp, B) \sum_{i=1}^{n-1} t(i, A)(B^{kp})^i t(p, B^{kp}) \\ &= (a^p)^{t(p,\, b^p)} + kpb^{p^2} - c.(1 + B^p + \cdots + B^{p(k-1)}) \\ &\quad \times t(p, B)t(p, B^p) \sum_{i=1}^{p-1} t(i, A)(B^{kp})^i \\ &= (a^p)^{t(p,\, b^p)} + kpb^{p^2} - kc.t(p^2, B)l(p, A) \\ &= (a^p)^{t(p,\, b^p)} \in M_{21},\end{aligned}$$

and $0 \neq ((ab^{kp})^p)^{t(p,\, b)} \in M_{12_i}$.

LEMMA 4.1. *For* $1 \le i \le p-1$, *we have* $(b^p)^{t(p,\, a^i)} \in M_{22} \oplus M_{12_k}$, *where* $ik \equiv 1 \bmod p$.

Proof. On writing additively, we have

$$(b^p)^{t(p,\, a^i)} = b^{p^2} - c.t(p, B)t(i, A) \sum_{j=1}^{p-1} t(j, A^i)B^{p(p-1-j)}.$$

If we show that $(b^p)^{t(p,\, a^i)(a-1)(a-b^{kp})} = 0$, then $(b^p)^{t(p,\, a^i)} \in M_{21} \oplus M_{22} \oplus M_{12_k}$ would follow. First we calculate

$$\begin{aligned}(b^p)^{t(p,\, a^i)(a-1)} &= (b^{p^2})^{a-1} - c.t(p, B)(A^i - 1) \sum_{j=1}^{p-1} t(j, A^i)B^{p(p-1-j)} \\ &= -c.t(p^2, B) - c.t(p, B) \sum_{j=1}^{p-1} (A^{ij} - 1)B^{p(p-1-j)} \\ &= -c.t(p^2, B) + c.t(p, B)t(p, B^p) \\ &\quad - c.t(p, B) \sum_{j=0}^{p-1} A^{ij}B^{p(p-1-j)} \\ &= -c.t(p, B) \sum_{j=0}^{p-1} A^{ij}B^{p(p-1-j)}.\end{aligned}$$

Now we write $s = p - 1 - j$. Then

$$
\begin{aligned}
(b^p)^{t(p,\, a^i)(a-1)(a-b^{kp})} &= -c.t(p, B)(A - B^{kp}) \sum_{j=0}^{p-1} A^{ij} B^{p(p-1-j)} \\
&= -c.t(p, B) \sum_{s=0}^{p-1} A^{i(p-1-s)+1} B^{ps} \\
&\quad + c.t(p, B) \sum_{s=0}^{p-1} A^{i(p-1-s)} B^{p(s+k)},
\end{aligned}
$$

and after reordering the terms of $c.t(p, B) \sum_{s=0}^{p-1} A^{i(p-1-s)} B^{p(s+k)}$, we have

$$
\begin{aligned}
c.t(p, B) \sum_{s=0}^{p-1} A^{i(p-1-s)} B^{p(s+k)} &= c.t(p, B) \sum_{s=0}^{p-1} A^{i(p-1-s)} B^{p(s+k)} \\
&= c.t(p, B) \sum_{s=0}^{p-1} A^{i(p-1-s+k)} B^{ps} \\
&= c.t(p, B) \sum_{s=0}^{p-1} A^{i(p-1-s)+1} B^{ps},
\end{aligned}
$$

since $ik \equiv 1 \bmod p$. Thus $(b^p)^{t(p,a^i)(a-1)(a-b^{kp})} = 0$ and

$$(b^p)^{t(p,\, a^i)} \in M_{21} \oplus M_{22} \oplus M_{12_k}.$$

To prove that $(b^p)^{t(p,\, a^i)} \in M_{22} \oplus M_{12_k}$, it is enough to show that $(b^p)^{t(p,\, a^i)t(p,\, b^p)} = 0$. Then

$$
\begin{aligned}
(b^p)^{t(p,\, a^i)t(p,\, b^p)} &= pb^{p^2} - c.t(p, B)t(p, B^p)t(i, A) \sum_{j=1}^{p-1} t(j, A^i) B^{p(p-1-j)} \\
&= pb^{p^2} - c.t(p^2, B)t(i, A) \sum_{j=1}^{p-1} t(j, A^i) \\
&= pb^{p^2} - c.t(p^2, B)t(i, A)l(p, A^i).
\end{aligned}
$$

Now we have

$$c.l(p, A^i)(A^i - 1) = c.(t(p, A^i) - p) = c.(t(p, A) - p) = c.d(A)l(p, A)$$

and therefore

$$c.(l(p, A^i)t(i, A) - l(p, A))d(A) = 0,$$

and $c.l(p, A^i)t(i, A) - c.l(p, A) \in M_2$. Thus

$$c.l(p, A^i)t(i, A) = c.l(p, A) + m_2,$$

where $m_2 \in M_2$. However, since $m_2.t(p^2, B) = 0$, we have

$$\begin{aligned}(b^p)^{t(p, a^i)t(p, b^p)} &= pb^{p^2} - c.t(p^2, B)t(i, A)l(p, A^i) \\ &= pb^{p^2} - c.t(p^2, B)l(p, A) = 0,\end{aligned}$$

and therefore, for $1 \le i \le p-1$, we have $(b^p)^{t(p, a^i)} \in M_{22} \oplus M_{12_k}$, where $ik \equiv 1 \bmod p$. Furthermore, we can easily verify that the components of it in both submodules are non-trivial. ∎

PROPOSITION 4.2. *The group* $K(p, p^2)$ *has* $\frac{2^p - 2}{p} + 2$ *proper, non-isomorphic torsion-free quotients.*

Proof. Let $N \trianglelefteq G = K(p, p^2)$. Then $R = \mathbb{Q} \otimes (N \cap V)$ is the sum of some of the $2p+1$ submodules obtained in the decomposition of M. Therefore we have $2^{2p+1} - 1$ cases to study (we exclude the trivial case). It follows from Lemma 2.2 that, for any possibility for R being studied, there will be at most one possible $N \trianglelefteq K(p, p^2)$, such that $\mathbb{Q} \otimes (N \cap V) = R$ and $K(p, p^2)/N$ is torsion-free.

Since $b^{p^2} \in M_3$ and $(ab^k)^{p^2} \in M_{11_i}$, where $k + i = p^2$, we have that $M_3, M_{11_i} \not\subseteq R$. Thus we have $2^{p+1} - 1$ cases to study. If $M_{21} \subseteq R$, it follows from Lemma 3.1 that no other submodule of M can be contained in R.

Let $N = \langle (a^p)^{t(p, b^p)} \rangle^G$. It is clear that $\mathbb{Q} \otimes N = M_{21}$. Now, in Proposition 3.2 we showed that

$$\frac{G}{N} \cong \left\langle a, b \mid (a^p)^{t(p, b^p)}, (b^{p^2})^{t(p, a)}, [[a, b], a^p], \left[[a, b], b^{p^2}\right], \ \textit{metabelian} \right\rangle$$

is a Bieberbach group of dimension $p^3 - 1 - p + 1 = p^3 - p$, with point-group isomorphic to $C_p \times C_{p^2}$ and commutator quotient $C_{p^2} \times C_{p^3}$. Notice that this group has no proper torsion-free quotients. We denote it by T_M.

Now R can be equal to the sum of any of the submodules M_{12_j} and M_{22}. Thus we have $2^p - 1$ cases to study. Once we find $N \trianglelefteq V$, such that $\mathbb{Q} \otimes N = R$ and $\frac{V}{N}$ is torsion-free, that must be enough, since it follows from Lemma 2.2 and the remark before Proposition 3.3 that the group $\frac{G}{N}$ will have a quotient isomorphic to $K(p, p)$, with the kernel of the epimorphism contained in $\frac{V}{N}$.

Let R be equal to one of these submodules, for instance M_{22}. If $N = \langle (a^p)^{t(p, b)} \rangle^G$, then $\mathbb{Q} \otimes N = M_{22}$ and if we compute the Smith normal form for the matrix of generators of $\frac{V}{N}$, we can show, in a manner similar to the proof of Proposition 3.2, that $\frac{V}{N}$ is torsion-free. Thus

$$\frac{G}{N} \cong \left\langle a, b \mid (a^p)^{t(p, b)}, \left(b^{p^2}\right)^{t(p, a)}, [[a, b], a^p], \left[[a, b], b^{p^2}\right], \ \textit{metabelian} \right\rangle$$

is a Bieberbach group of dimension $p^3 - 1 - p^2 + p = (p^2+1)(p-1)$. It also has point-group isomorphic to $C_p \times C_{p^2}$ and commutator quotient isomorphic to $C_{p^2} \times C_{p^3}$. All the remaining cases for R being equal to one of the submodules M_{12_j}, M_{22} is isomorphic to the group above, by the isomorphism induced by the automorphism of $K(p, p^2)$ given by $a \mapsto ab^p$, $b \mapsto b$; see [3]. We denote this group by T_1.

Now suppose R is sum of two of the submodules M_{12_j}, M_{22}. If $p = 2$, then $\frac{G}{N} \cong K(2,2)$. If p is odd, then we have $\binom{p}{2}$ possibilities in this case, but using once more the isomorphism defined above, we can suppose that $M_{22} \subseteq R$ and we have $\frac{1}{p}\binom{p}{2} = \frac{p-1}{2}$ cases to study. We have seen that

$$(b^p)^{t(p,\, a^i)} \in M_{22} \oplus M_{12_k},$$

where $ik \equiv 1 \bmod p$. Let $N_k = \langle (a^p)^{t(p,\, b)}, (b^p)^{t(p,\, a^i)} \rangle^G$, for $1 \le k \le \frac{p-1}{2}$. We can show again that $\frac{V}{N_k}$ is torsion-free, since N_k is a pure submodule of V. And because $\frac{G}{N_k}$ has a quotient isomorphic to $K(p, p)$, with the kernel contained in $\frac{V}{N_k}$, we have that

$$\frac{G}{N_k} \cong \Big\langle a, b \mid (a^p)^{t(p,\, b)}, (b^p)^{t(p,\, a^i)}, \big(b^{p^2}\big)^{t(p,\, a)}, \\ [[a,b], a^p], \big[[a,b], b^{p^2}\big],\ \textit{metabelian} \Big\rangle$$

is a Bieberbach group of dimension $p^3 - 1 - 2(p^2 - p)$, with point-group isomorphic to $C_p \times C_{p^2}$ and commutator quotient $C_{p^2} \times C_{p^2}$. There are $\frac{p-1}{2}$ groups and applying Theorem 2.2, Chapter III of [2], we can show that they are all non-isomorphic, since there is not a semi-linear homomorphism (f, σ) between their translation subgroups, such that $f(m.A^iB^j) = f(m).\sigma(A^iB^j)$. We denote these groups by $T_{21}, T_{22}, \ldots, T_{2i_2}$, where $i_2 = \frac{p-1}{2}$.

If R is equal to the sum of n submodules, $3 \le n \le p-1$, then using the automorphism defined above, we can suppose that $M_{22} \subseteq R$ and there are $\frac{1}{p}\binom{p}{n} = i_n$ non-isomorphic torsion-free quotients (using again Theorem 2.2, Chapter III of [2]), defined as follows:

For each n, we obtain R_k, $1 \le k \le i_n$, and define $N_k = V \cap R_k$. Then N_k is a pure submodule of V and $\frac{V}{N_k}$ is torsion-free. Since $\frac{G}{N_k}$ has quotient isomorphic to $K(p, p)$, with kernel contained in $\frac{V}{N_k}$, we have that $\frac{G}{N_k}$ is a Bieberbach group, of dimension $p^3 - 1 - n(p^2 - p)$. Furthermore, G/K_k has point-group isomorphic to $C_p \times C_{p^2}$ (otherwise it would be isomorphic to $K(p, p)$) and commutator quotient isomorphic to $C_{p^2} \times C_{p^2}$. Indeed, they are all quotients of some of the T_{2j} defined above and have $K(p, p)$ as quotient. And of course these groups have commutator quotient isomorphic to $C_{p^2} \times C_{p^2}$. For each n, we have $i_n = \frac{1}{p}\binom{p}{n}$ quotients, which we denote by $T_{n1}, \ldots, T_{ni_n}$.

And finally, if R is the sum of p submodules M_{22}, M_{12_j}, we have $\frac{G}{N}$ isomorphic to $K(p, p)$. Thus we have a total of $\frac{2p-2}{p} + 2$ proper, non-isomorphic quotients of $K(p, p^2)$. Notice that T_M and $K(p, p)$ are the only ones that have no proper torsion-free quotient. ∎

In particular, when $p = 2$, the group $K(2, 4)$ has 3 proper, non-isomorphic torsion-free quotients, given by

$$\begin{aligned}
H_1 &= \langle a, b \mid (a^2)^{1+b^2}, (b^4)^{1+a}, [[a, b], b^4], [[a, b], a^2],\ \textit{metabelian}\rangle \\
H_2 &= \langle a, b \mid (a^2)^{1+b}, (b^4)^{1+a}, [[a, b], b^4], [[a, b], a^2],\ \textit{metabelian}\rangle \\
K(2, 2) &= \langle a, b \mid (a^2)^{1+b}, (b^2)^{1+a}, [[a, b], b^2], [[a, b], a^2],\ \textit{metabelian}\rangle,
\end{aligned}$$

where H_1 and H_2 have dimension 6 and 5, respectively. Both have point-group isomorphic to $C_2 \times C_4$ and commutator quotient $C_4 \times C_8$.

We should notice that even though, for p odd, we found torsion-free quotients with point-group $C_p \times C_{p^2}$ and commutator quotient $C_{p^2} \times C_{p^2}$, this did not happen when $p = 2$.

5. TORSION-FREE QUOTIENTS OF $K(2, 8)$ AND $K(4, 4)$

Using the method of the previous section, we are able to produce the following complete list of torsion-free quotients of $K(2, 8)$ and $K(4, 4)$; the proof can be found in [4].

PROPOSITION 4.4 OF [4]. *The group $K(2, 8)$ has 12 proper, non-isomorphic torsion-free quotients*,

$$\begin{aligned}
Q_1 &= \langle a, b \mid (a^2)^{(1+b^2)(1+b^4)}, (b^8)^{t(2, a)}, [[a, b], b^8], [[a, b], a^2],\ \textit{metabelian}\rangle \\
Q_2 &= \langle a, b \mid (a^2)^{(1+b)(1+b^4)}, (b^8)^{t(2, a)}, [[a, b], b^8], [[a, b], a^2],\ \textit{metabelian}\rangle \\
Q_3 &= \langle a, b \mid (a^2)^{t(8, b)}, (b^8)^{t(2, a)}, ((ab^2)^4)^{1+b}, [[a, b], b^8], \\
&\qquad [[a, b], a^2],\ \textit{metabelian}\rangle \\
Q_4 &= \langle a, b \mid (a^2)^{t(4, b)}, (b^8)^{t(2, a)}, [[a, b], b^8], [[a, b], a^2],\ \textit{metabelian}\rangle \\
Q_5 &= \langle a, b \mid (a^2)^{1+b}, (b^8)^{t(2, a)}, [[a, b], b^8], [[a, b], a^2],\ \textit{metabelian}\rangle \\
Q_6 &= \langle a, b \mid (a^2)^{1+b^2}, (b^8)^{t(2, a)}, [[a, b], b^8], [[a, b], a^2],\ \textit{metabelian}\rangle \\
Q_7 &= \langle a, b \mid (a^2)^{1+b^4}, (b^8)^{t(2, a)}, [[a, b], b^8], [[a, b], a^2],\ \textit{metabelian}\rangle \\
Q_8 &= \langle a, b \mid (a^2)^{t(4, b)}, (b^8)^{t(2, a)}, ((ab^2)^4)^{1+b}, b^8[a, b]^{(1+b)(a-b^4)}, \\
&\qquad [[a, b], b^8], [[a, b], a^2],\ \textit{metabelian}\rangle
\end{aligned}$$

$$Q_9 = K(2,4)$$
$$Q_{10} = H_1 = \langle a, b \mid (a^2)^{1+b^2}, (b^4)^{t(2,a)}, [[a,b], b^4], [[a,b], a^2], \textit{metabelian}\rangle$$
$$Q_{11} = H_2 = \langle a, b \mid (a^2)^{1+b}, (b^4)^{t(2,a)}, [[a,b], b^4], [[a,b], a^2], \textit{metabelian}\rangle$$
$$Q_{12} = K(2,2).$$

The groups Q_1, Q_2, Q_3, and Q_4 have point-group isomorphic to $C_2 \times C_8$ and commutator quotient $C_8 \times C_{16}$, with dimensions 14, 13, 13, and 11, respectively.

The groups Q_5, Q_6, and Q_7 have point-group isomorphic to $C_2 \times C_8$ and commutator quotient $C_4 \times C_{16}$, with dimensions 9, 10, and 12, respectively.

The group Q_8 has point-group isomorphic to $C_2 \times C_8$, commutator quotient $C_8 \times C_8$, and dimension 9.

The groups Q_9, Q_{10}, Q_{11}, and Q_{12} are quotients of $K(2,4)$ and have already been described.

It follows from the lattice of $\Lambda_{2,1,3}$ (Fig. 1) that Q_7, H_1, and $K(2,2)$ have no proper torsion-free quotients.

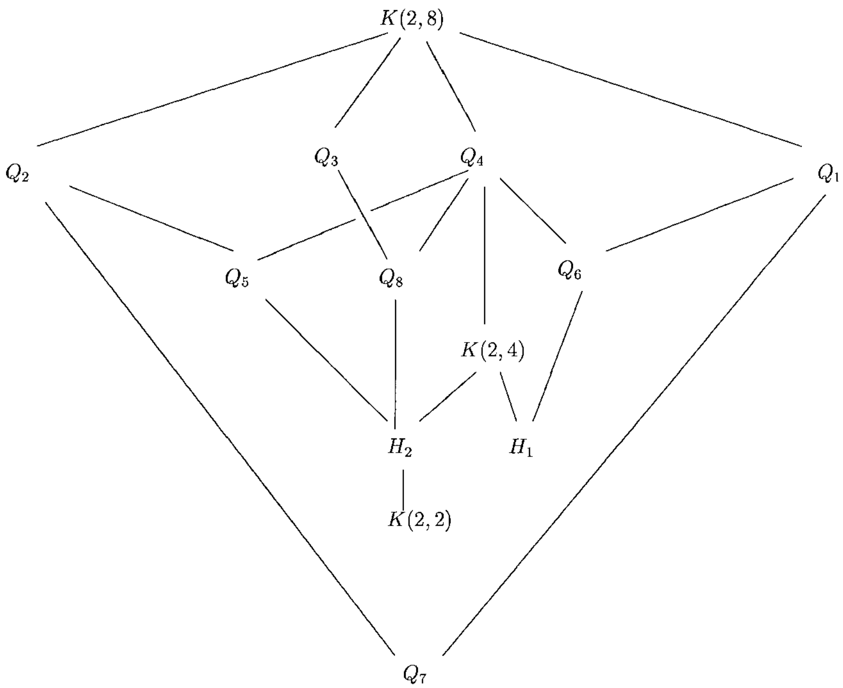

FIG. 1. $\Lambda_{2,1,3}$.

PROPOSITION 4.5 OF [4]. *The group $K(4,4)$ has* 19 *proper, non-isomorphic torsion-free quotients*,

$$S_1 = \langle a, b \mid (a^4)^{1+b^2}, (b^4)^{t(4,a)}, [[a,b],b^4], [[a,b],a^4], \textit{ metabelian}\rangle$$

$$S_2 = \langle a, b \mid (a^4)^{1+b}, (b^4)^{t(4,a)}, [[a,b],b^4], [[a,b],a^4], \textit{ metabelian}\rangle$$

$$S_3 = \langle a, b \mid (a^4)^{1+b^2}, (b^4)^{1+a^2}, [[a,b],b^4], [[a,b],a^4], \textit{ metabelian}\rangle$$

$$S_4 = \langle a, b \mid (a^4)^{1+b^2}, (b^4)^{1+a}, [[a,b],b^4], [[a,b],a^4], \textit{ metabelian}\rangle$$

$$S_5 = \langle a, b \mid (a^4)^{1+b}, (b^4)^{1+a}, [[a,b],b^4], [[a,b],a^4], \textit{ metabelian}\rangle$$

$$\begin{aligned} S_6 = \langle a, b \mid\ & (b^4)^{1+a}, ((a^2b)^4)^{1+a}, (a^2)^{(1+b)\cdot(1+b^2)}, [[a,b],b^4], \\ & [[a,b],a^4], \textit{ metabelian}\rangle \end{aligned}$$

$$\begin{aligned} S_7 = \langle a, b \mid\ & (a^4)^{1+b^2}, (b^4)^{1+a^2}, ((ab)^4)^{1+a^2}, [[a,b],b^4], \\ & [[a,b],a^4], \textit{ metabelian}\rangle \end{aligned}$$

$$\begin{aligned} S_8 = \langle a, b \mid\ & (a^4)^{1+b^2}, (b^4)^{1+a^2}, ((ab)^4)^{1+a}, [[a,b],b^4], \\ & [[a,b],a^4], \textit{ metabelian}\rangle \end{aligned}$$

$$\begin{aligned} S_9 = \langle a, b \mid\ & (a^4)^{1+b^2}, (b^4)^{1+a}, ((a^2b)^4)^{1+a}, (a^2)^{(1+b)\cdot(1+b^2)}, [[a,b],b^4], \\ & [[a,b],a^4], \textit{ metabelian}\rangle \end{aligned}$$

$$\begin{aligned} S_{10} = \langle a, b \mid\ & (a^4)^{1+b^2}, (b^4)^{1+a}, ((ab)^4)^{1+a}, [[a,b],b^4], \\ & [[a,b],a^4], \textit{ metabelian}\rangle \end{aligned}$$

$$\begin{aligned} S_{11} = \langle a, b \mid\ & (a^4)^{1+b}, (b^4)^{1+a}, ((a^2b)^4)^{1+a}, (a^2)^{(1+b)\cdot(1+b^2)}, [[a,b],b^4], \\ & [[a,b],a^4], \textit{ metabelian}\rangle \end{aligned}$$

$$\begin{aligned} S_{12} = \langle a, b \mid\ & (a^4)^{1+b}, (b^4)^{1+a}, ((ab)^4)^{1+a}, [[a,b],b^4], \\ & [[a,b],a^4], \textit{ metabelian}\rangle \end{aligned}$$

$$\begin{aligned} S_{13} = \langle a, b \mid\ & (a^4)^{1+b}, (b^4)^{1+a}, ((ab^3)^4)^{1+a}, [[a,b],b^4], \\ & [[a,b],a^4], \textit{ metabelian}\rangle \end{aligned}$$

$$\begin{aligned} S_{14} = \langle a, b \mid\ & (a^4)^{1+b}, ((ab)^4)^{1+b}, (b^4)^{1+a}, ((a^2b)^4)^{1+a}, (a^2)^{(1+b)\cdot(1+b^2)}, \\ & [a,b]^{1+b+a^3+ab}, [[a,b],b^4], [[a,b],a^4], \textit{ metabelian}\rangle \end{aligned}$$

$$\begin{aligned} S_{15} = \langle a, b \mid\ & (a^2)^{1+b^2}, (a^4)^{t(4,b)}, (b^4)^{1+a}, [[a,b],b^4], \\ & [[a,b],a^4], \textit{ metabelian}\rangle \end{aligned}$$

$$\begin{aligned} S_{16} = K(2,4) = \langle a, b \mid\ & (a^2)^{t(4,b)}, (b^4)^{1+a}, [[a,b],b^4], \\ & [[a,b],a^2], \textit{ metabelian}\rangle \end{aligned}$$

$$S_{17} = H_1 = \langle a, b \mid (a^2)^{1+b^2}, (b^4)^{1+a}, [[a,b],b^4], [[a,b],a^2], \textit{metabelian}\rangle$$

$$S_{18} = H_2 = \langle a, b \mid (a^2)^{1+b}, (b^4)^{1+a}, [[a,b],b^4], [[a,b],a^2], \textit{metabelian}\rangle$$

$$S_{19} = K(2,2) = \langle a, b \mid (a^2)^{1+b}, (b^2)^{1+a}, [[a,b],b^2], [[a,b],a^2], \textit{metabelian}\rangle.$$

The groups S_1 and S_2 have point-group $C_4 \times C_4$, commutator quotient $C_8 \times C_{16}$, and dimensions 14 and 13, respectively.

The groups S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10}, S_{11}, S_{12}, S_{13}, and S_{14} all have point-group $C_4 \times C_4$ and commutator quotient $C_8 \times C_8$, with dimensions 13, 12, 11, 11, 12, 11, 10, 10, 9, 9, 9, and 7, respectively.

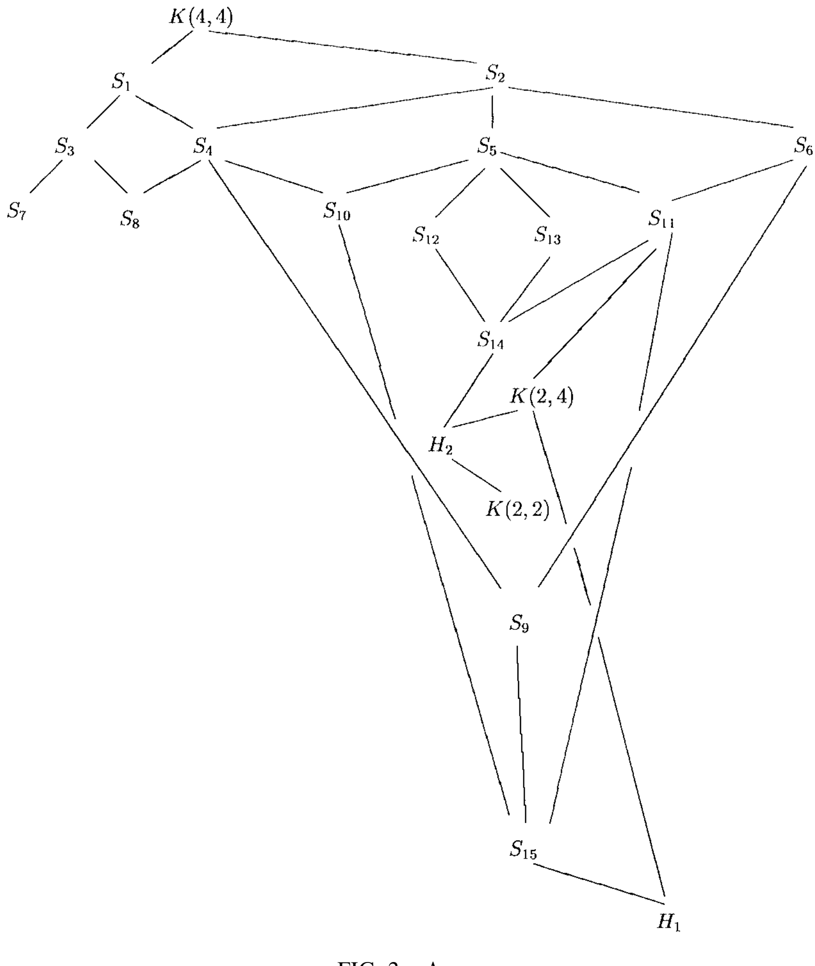

FIG. 2. $\Lambda_{2,2,2}$.

The group S_{15} has point-group isomorphic to $C_4 \times C_4$, commutator quotient $C_4 \times C_8$, and dimension 8.

The groups S_{16}, S_{17}, S_{18}, and S_{19} are quotients of $K(2, 4)$ and have already been described.

It follows from the lattice of $\Lambda_{2,2,2}$ (Fig. 2) that the groups S_7, S_8, H_1, and $K(2, 2)$ have no proper torsion-free quotients.

From the list of quotients of $K(4, 4)$, we can obtain the following characterization of $K(2, 2)$.

THEOREM B. *Let G be a finitely generated, torsion-free metabelian group, with commutator quotient isomorphic to $C_4 \times C_4$. Then G is isomorphic to*

$$K(2,2) = \langle a, b \mid (a^2)^{1+b}, (b^2)^{1+a}, [[a,b], a^2], [[a,b], b^2], \textit{metabelian} \rangle,$$

the fundamental group of the Hantzsche–Wendt manifold.

Proof. Let $a, b \in G$ be the generators of G modulo G' and $H = \langle a, b \rangle$. Then $G = HG'$ and it follows from Theorem 2 of [5] that H is a 2-generated torsion-free metabelian group, with

$$\frac{H}{H'} \cong \frac{G}{G'} \cong C_4 \times C_4.$$

Furthermore, both G and H are Bieberbach groups. We denote by V_H the translation subgroup of H. Since $H' \leq V_H$, it follows from Theorem A of [3] that H is isomorphic to a torsion-free quotient of $K(4, 4)$. Now, by the list of torsion-free quotients of $K(4, 4)$ given above, the only torsion-free quotient of $K(4, 4)$ with commutator quotient isomorphic to $C_4 \times C_4$ is $K(2, 2)$. Thus $H \cong K(2, 2)$.

Furthermore, we can repeat part of the proof of Proposition 2.3 of [3] and show that $G' = [G', H]H'$. Then we define the normal subgroup $N = (G')^2 H'$, and since G is finitely generated, we have that $\frac{G}{N}$ is a finite 2-group. Now we can compute the second and third terms of the lower central series of $\frac{G}{N}$,

$$\Gamma_2\left(\frac{G}{N}\right) = \left[\frac{G}{N}, \frac{G}{N}\right] = \frac{G'N}{N} = \frac{G'}{N}$$

and

$$\Gamma_3\left(\frac{G}{N}\right) = \left[\frac{G'}{N}, \frac{G}{N}\right] = \frac{[G', G]N}{N} = \frac{[G', G'H]N}{N} = \frac{[G', H]H'(G')^2}{N} = \frac{G'}{N}.$$

Thus $\Gamma_2(\frac{G}{N}) = \Gamma_3(\frac{G}{N})$, and because $\frac{G}{N}$ is nilpotent, $G' = N = (G')^2 H'$. Now we can show that

$$\dim(H) = rk(H') = rk(G') = \dim(G).$$

Thus G is also a 3-dimensional Bieberbach group, with commutator quotient isomorphic to $C_4 \times C_4$. By [1], we have that G is isomorphic to the fundamental group of the Hantzsche–Wendt manifold; that is, $G \cong K(2, 2)$. ∎

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